

An Epistemological Derivation of Quantum Logic

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February 1, 2008

Abstract

This paper deals with the foundations of quantum mechanics. We start by outlining the characterisation, due to Birkhoff and Von Neumann, of the logical structures of the theories of classical physics and quantum mechanics, as boolean and modular lattices respectively. We then derive these descriptions from what we claim are basic properties of any physical theory - i.e. the notion that a quantity in such a theory may be analysed into parts and that the results of this analysis may be treated in languages with an underlying boolean structure. We shall see that in the course of constructing a model of a theory with these properties different indistinguishable possibilities will arise for how the elements of the model may be named, that is to say different possibilities arise for how they can be associated with points from Set. Taking a particular collection of possibilities gives the usual boolean lattice of the propositions of classical physics. Taking all possibilities - in a sense, the set of all things that may be described by physical theories - gives the lattice of quantum mechanical propositions. This gives an interpretation of quantum mechanics as the complete set of such possible descriptions, the complete physical description of the world.

1 Introduction

In physics, the theory of statics comprises propositions about two basic observables, position and momentum. These propositions define ranges of values for each observable.

We may then add to this a notion of phase space and a law of propagation associated with the physical system.

In a classical physical theory the subsets of phase space are correlated with propositions about the ranges of values so that there is an obvious correspondence between the set theoretic operators of union and intersection and the logical connectives ‘and’ and ‘or’ - with set theoretic inclusion corresponding to logical implication.

In a quantum mechanical system the propositions concerning ranges of the observables are correlated with the subspaces of a Hilbert space and the logical connectives between propositions correspond to set products, sums and complements with, once again, inclusion corresponding to implication.

Obviously therefore the quantum mechanical and classical propositional calculi differ from an algebraic standpoint. This difference is precisely the following. In the classical propositional calculus the logical connectives between propositions obey the distributive law,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

while in the propositional calculus of quantum mechanics \wedge and \vee do not obey this law although they obey a modified version of it - the weaker modular law,

$$\text{if } a \leq c \text{ then } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

In both cases \wedge and \vee obey the other usual assumptions sufficient to make “ $a \leq b$ iff $a \wedge b = a$ (or equivalently $a \vee b = b$) ” a partial order.

If we assume in addition to the modular law that every proposition can be written as the union of basic elements – atomicity – and any pair of these basic elements have a common complement – perspectivity – then we get the characterisation due to Birkhoff and Von Neumann [1] of the lattice of quantum mechanical propositions as an infinite, modular, atomic, perspective lattice.

Since all the nonclassical results of quantum mechanics arise from this distinction in algebraic structure many attempts have been made to explain the need for this modular law.

Here it is derived from what we take to be the basic requirements demanded of any physical theory — that the theory contains an operator representing the analysis of quantities into distinct named parts — and that we can treat this operator in boolean languages i.e. we can identify substructures produced by the analysis operator with sublattices of boolean lattices subject to certain consistency requirements.

To produce a model points from some model of set theory, *Set*, must be assigned to the variables of the theory. We have not specified which points in particular must be chosen from *Set* and certain indistinguishable possibilities arise for models generated in this way. Choosing a single set

of possibilities gives the usual boolean lattice as we would expect but we show that the quantum mechanical lattice is the lattice generated by taking all consistent possibilities. Quantum mechanics is therefore in a sense the fullest description of the physical world if the world is restricted to what can be modelled by structures of the above kind.

2 Definitions

We have stated that the propositions of classical physics are points in boolean lattices. How is such a lattice defined ?

A structure will be called a lattice iff to any pair of its elements x and y there correspond elements $x \wedge y$, $x \vee y$ with the operators \wedge, \vee ('join' and 'meet') satisfying

$$x \wedge x = x, \quad x \vee x = x, \quad (\text{idempotency}) \quad (1)$$

$$x \wedge y = y \wedge x, \quad x \vee y = y \vee x, \quad (\text{commutativity}) \quad (2)$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z, \quad (\text{associativity}) \quad (3)$$

$$x \wedge (x \vee y) = x \vee (x \wedge y) = x, \quad (\text{absorbtion}) \quad (4)$$

\wedge, \vee then also define a partial order given by $x \leq y$ if $x \wedge y = x$ (or, equivalently, $x \vee y = y$).

If in addition \wedge, \vee satisfy

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad (5)$$

the lattice is said to be distributive.

If instead they satisfy the weaker

$$x \leq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad (6)$$

the lattice is said to be modular.

The following is an equivalent formulation of the modular law for finite lattices. Let B be a lattice and h a positive function, the height function, defined on B where for all a, b, c , in B

$$h(a \wedge b) = h(a) + h(b) - h(a \vee b) \quad (7)$$

Then B satisfies the modular law. Clearly a distributive lattice supports a function satisfying (7).

If there exist elements $0, 1$ in the lattice satisfying

$$0 \leq x \leq 1 \quad \forall x$$

and if for every element x in the lattice there is a y with

$$x \wedge y = 0, \quad x \vee y = 1$$

the lattice is said to be complemented.

A complemented distributive lattice is called a boolean lattice.

For x, y elements of a lattice, if $\exists z$ with

$$\begin{aligned} x \wedge z = 0, \quad x \vee z = 1 \\ y \wedge z = 0, \quad y \vee z = 1 \end{aligned}$$

then x and y are said to have a common complement z . Such x and y are said to be perspective. If every pair of points in a lattice are perspective the lattice is said to be perspective.

Finally, an atom is a point x such that $\forall y$ in the lattice

$$y < x \Rightarrow y = 0$$

3 Outline of the Argument

If we want to produce an analytic description of the world then, formally speaking, it must at least satisfy the conditions sketched at the end of section 1 above, which we restate here. Our central premise is that anything we think of as a physical theory must incorporate the idea of analysis of individuals into parts and the treatment as far as is consistently possible of these parts in languages containing (boolean) joins and meets and that any attempt to construct a model of such a theory will naturally reflect this. We can think of the physical theory as containing the collection of statements arising from analysing the universe into parts through measurement and the further statements that can be made about these parts in the boolean languages that underly any formal discussion.

We first give a lengthier informal description of the properties that should be satisfied by a structure representing a physical theory and later define them precisely.

Firstly, the structure should be nonempty – it should contain two points chosen from set theory call them 0 and 1 , and, further, in the ordering relation we will define on the structure, 0 and 1 satisfy $0 \leq x \leq 1 \quad \forall x$ in the structure.

Secondly, a structure representing a physical theory should support an operator representing the analysis or division of an element into parts. Given any element a in the structure the structure should also contain an element b_a distinct from a , with $a = b_a \vee a$ contained in the structure and given such a b_a there should also exist a c_a in the structure with $a = b_a \vee c_a$ and such that $\forall e \ c_a \vee e = c_a$ and $b_a \vee e = b_a$ iff $e = 0$. We claim that to represent the natural notion of analysis of a whole into parts \vee should be a binary operator generating a partial order.

It is important to note here that in terms of using this property to generate such a structure the elements from *Set* represented by b_a and c_a , above are not uniquely specified — there are many possible candidates in *Set* for such elements.

A chain between a_1 and a_h in the structure is a set of distinct points $\{a_1, \dots, a_h\}$ with $\forall i \ a_i \vee a_{i+1} = a_i$. We claim that to represent the idea of analysis \vee should satisfy a further condition. Let a refinement of a chain be a larger chain containing it. Then refinements exist subject to the following. For a given a, b in the structure either any path between a and b may be refined or there exists a bound $d(a, b)$ such that only those paths of length $< d(a, b)$ may be refined.

We claim that if in addition \vee is consistent with the existence of a lattice extending it and satisfying the same properties then this operator represents our natural notion of analysis i.e. the operator represents the ‘part of’ relation arising from measurement in a physical theory.

Thirdly, these two principles give rise to many distinct elements and statements of the form $a = b \vee c$ where a , b and c represent points from *Set*. Let M be a set of statements generated in this way. While \vee is consistent with the existence of a lattice with the given properties it does not generate it. In some cases it may be possible to embed M in a natural language in such a way that the boolean operators of the language generate the lattice. However it may also be that the lattice we require may not be embeddable in such a language. This is the source of the difficulties in the interpretation of quantum mechanics. All such difficulties can be reduced to the problem of trying to construct the above lattice in terms of an analysis based on what is constructible in boolean lattices. The purpose of this paper is to produce such an analysis.

As we have just said it may be possible to make statements about M in a natural language, that is to say it may be possible to embed M in a boolean lattice in such a way that the combined structure still satisfies the above requirements on an analytic operator. A physical theory should include the

expansion of the analytic operator by statements which can be built up by a treatment of the operator in a natural language subject to a general demand of consistency.

So a physical theory can make statements about the analytic operator in a language, L , at least strong enough to contain the statements that could arise in any natural discussion, i.e. L contains the boolean operators \wedge_L , \vee_L and $'_L$ or complement. Consider the statements which we may be able to make about M in such a language. It may be that there is a structure generated from M using \wedge_L , \vee_L and $'_L$ extending M to a boolean lattice B_M in such a way that B_M agrees with the lattice operators, height function and possibly the 0, 1 of M where they overlap and – restricting ourselves to extending the structure as a partition operator – such that the combined structure is still capable of extension to a lattice satisfying the properties described above.

One obvious requirement that this extension principle be a consistent one is that we allow an extension only when it is consistent with all such extensions of the substructures of M . For a given M there may be a number of ways of forming boolean extensions of the substructures of M , call them M_i , in such a way that they are mutually consistent and consistent with the other conditions outlined above. A collection $\{B_{M_i}\}$ of such extensions may be maximal i.e. there is no additional substructure M_j such that any B_{M_j} is consistent with the collection $\{B_{M_i}\}$. Call such a collection a cover of M . Then the existence of any such cover will imply the existence of a boolean extension of M , B_M , iff every cover of M contains such a B_M . The third property that should be enjoyed by a physical theory is that if this condition holds the structure may be extended by some B_M . We are allowing M and its subsets to be extended by the boolean operators of the language only if this happens in a consistent way.

We will show that these properties are realised in a boolean lattice and any structure given by these properties contains such a lattice.

In describing these properties we alluded to the fact that points chosen to name variables in the theory are not uniquely specified. It turns out we can generate a structure realising all of these possible choices simultaneously by first extending the second property to state that in addition to b_a, c_a with $b_a \vee c_a = a$ there is also a point $b'_a \neq b_a, c_a$ with $b'_a \vee d_a = a$.

The structure which is generated by extending Principle II in this way is P_{n-1} , Birkhoff and Von Neumann's characterisation of the quantum mechanical lattice.

4 Construction Principles

Let $Const = \{c_1, \dots\}$ where c_i are distinct points chosen from some model of set theory, Set . Equivalently we may define the c_i to be distinct elements in the language and $\forall c_i, c_j, c_i \neq c_j$ is true.

In the last section we sketched a set of properties which a structure representing a set of physical propositions should have. We shall see that these properties are sufficient to represent such a set i.e. if we recast them as construction principles the structure they describe is that of the lattice of physical propositions. We describe these principles below.

I The structure should be nonempty.

PRINCIPLE I *The structure should contain a substructure consisting of two points chosen from $Const$. Call the points 0 and 1. We define $0 \vee 1 = 1$ and in all that follows it is consistent that $0 \vee x = x$, $x \vee 1 = 1$ for any x we construct.*

II From the principle of analysis of a whole into parts we get the following

PRINCIPLE II *Given any element a , in the structure the structure should also contain an element b_a , distinct from a chosen from $Const$, with the statement $a = b_a \vee a$ contained in the structure and given such an a and b_a the structure contains a c_a from $Const$ with $a = b_a \vee c_a$ and such that $\forall e c_a \vee e = c_a$ and $b_a \vee e = b_a$ iff $e = 0$.*

Whether b_a, c_a are points already chosen in the structure is not defined. It is important to note here that in extending the structure according to Principle II we have not specified which elements of $Const$ are represented by b_a and c_a . All that is demanded of them is that they should be distinct and different from a . Principle II states the existence of a condition to be satisfied by points from $Const$ without actually specifying those points.

We restate here that a chain between a_1 and a_h in the theory is a set of distinct points $\{a_1, \dots, a_h\}$ with $a_i \vee a_{i+1} = a_i \quad \forall i$. Let a refinement of a chain be a larger chain containing it.

Then Principle II extends the structure subject to refinements satisfying the following condition; for a given a, b in the structure either any path between a and b may be refined or there exists a bound $d(a, b)$ such that any path of length $< d(a, b)$ may be refined. Further \vee should satisfy the requirements on a partial order and be consistent with the existence of a lattice extending it and satisfying the requirements outlined above, i.e. .15in (i) Principle II holds in the lattice subject to Principle IV. .15in (ii) Refinements exist in the lattice subject to the condition that for any a and b in

the lattice there exists a bound $d(a, b)$ such that just those paths of length $\leq d(a, b)$ may be refined.

III Any natural language in which we might treat the analytic operator in Principle II contains the boolean operators. We extend the notion of a physical proposition to include statements we can make about this analytic principle in a natural language. Arising from this assumption the final principle is that the structure should include statements which are given by the possibility of extending substructures using the boolean operators of such languages. Let M be such a substructure and let \vee_M and \wedge_M as defined in M be consistent with the boolean property (5). Our natural notion of being able to treat the propositions of M in a rational language amounts to saying that they may be embedded in a structure equipped with boolean operators, \vee_L , \wedge_L and $'_L$, which are consistent with and extend the \vee and \wedge of the original structure in such a way that \vee and \wedge are consistent with the requirements described in the definition of Principle II. Thus the third principle generating the structure is defined as follows;

A substructure M comprises a set of statements of the form $a \vee b = c$, $a' \wedge b' = c'$... and statements regarding the function d , defined above, applied to points in M , $d(a, b) = d_{ab} \dots$

Let $\{M_i, i \in I\}$ be the set of substructures of M . For a given M_i it may be possible to define a structure B_{M_i} , a boolean lattice, on points chosen from $Const$, containing M_i and consistent with the structure as defined so far (where by consistency we mean that if we define a function h on the structure by $d(0, a) = h(a)$, and h_B is a height function on the distributive B_{M_i} , then $\wedge_B, \vee_B, h_B, 0_B$ and 1_B agree with $\wedge, \vee, h, 0$ and 1 , where defined in the structure so far, and the structure, extended by B_{M_i} , can still be extended to a lattice satisfying the requirements given above).

Next we define a cover of M . Given a collection of substructures of M , $\{M_i : i \in I\}$, suppose that for each M_i in the collection there exists such a boolean lattice and that these lattices are consistent in the sense described above with each other and the rest of the structure so far defined. If $\{B_{M_i} : i \in I\}$ is not contained in a larger collection of boolean lattices with this property $\{B_{M_i} : i \in I', I' \supset I\}$ we call $\{B_{M_i} : i \in I\}$ a cover of M . Such a cover represents a fullest possible mutually consistent treatment of the parts of M in boolean languages in the manner described.

If the existence of any such way of consistently extending parts of M to such lattices necessarily implies the existence of some B_M , i.e. if every cover of M contains a B_M , then we demand that the structure should be extended by some such B_M . In other words if granting that we can treat as many parts as possible of M in a rational language implies that we have a boolean lattice

containing M then we may add some such lattice to the structure. Recasting this as a construction principle we can say that the structure should contain that substructure common to all boolean lattices containing M and satisfying the consistency requirements described above.

PRINCIPLE III *For M a given substructure, if all maximal coverings, $\{B_{M_i} : i \in I, M_i \subset M\}$, contain a B_M , then the structure should contain a substructure common to all such B_M .*

We claim that this completely describes our natural notion of what can be said about the products of analysis in every boolean language.

In addition to these three principles we introduce an ad hoc assumption limiting the depth of the structure .

In their characterisation of the lattice of quantum mechanical propositions Birkhoff and Von Neumann, for the sake of simplifying the proof, restrict their attention to lattices of depth bounded by some $n \in \mathbf{N}$ - that is to say the length of every chain in the lattice is bounded by n . We do not need a restriction on the length of chains in the structures to prove the general result of this paper characterising the propositional calculi of the theories of classical and quantum mechanics. However, in the interest of simplifying our proof we too will assume this ad hoc bound on the length of chains in the structure and under this assumption derive from our generating principles Birkhoff and Von Neumann's restricted models of the propositional calculi. Again the structures we get in the absence of this ad hoc assumption are equivalent to the infinite models of Birkhoff and Von Neumann but the proof is more elaborate. We introduce the following

AD HOC PRINCIPLE IV *For some $n \in \mathbf{N}$ Principles II and III hold subject to the requirement that for any a in the structure $d(0, a) \leq n$*

Define $d(0, a) = h(a)$, called the height of a .

5 **These four principles generate and are realised in a boolean lattice.**

We now show that these four principles are realised in a boolean lattice of depth n and any structure generated by these four principles contains such a boolean lattice. Which is what we would expect to get from treating the products of an analytic principle like II in a language with an underlying boolean operator.

Let B be a model of a boolean lattice of depth n i.e. $h(1) = n$. We first show that B realises any structure constructed by I-IV, and hence these principles are consistent.

Principle I stating the existence of a 0 and 1 in B with $0 < x < 1 \forall x \in B$ is obviously satisfied in B .

Let C be a structure generated by Principles I-IV and let C be realised in B i.e. there exists a homomorphism f mapping $\{C\}$ into B and $\forall a, b, c \in C$ $a \wedge b = c \Rightarrow f(a) \wedge f(b) = f(c)$, $a \vee b = c \Rightarrow f(a) \vee f(b) = f(c)$, $h_C(a) = h_B(f(a))$. Then we will show that the extension of C by an application of Principle II is realised in B . Let a be a point in C realised in B . Then Principle II states that there should exist b_a and c_a with $b_a \vee c_a = a$ and such that $\forall e$ $c_a \vee e = c_a$ and $b_a \vee e = b_a$ iff $e = 0$ subject to this being consistent with the ad hoc Principle IV, i.e. subject to $h(a) \geq 2$. But if $h(a) \geq 2$ in B then obviously B contains such a b_a and c_a with $b_a \wedge c_a = 0$. Hence Principle II is satisfied in B .

Next we show that an extension of C by an application of Principle III is realised in B . Let M be a substructure of C . M is realised in B . Then Principle III asserts the existence of a substructure containing M common to all B_M . But B itself obviously realises such a substructure.

Since the extensions of C above were subject to the restrictions of Principle IV we are done.

We now show that Principles I-IV generate a model of B , the boolean lattice of depth n , and hence that any model generated by I-IV contains a submodel equivalent to B .

Let 1 be given by Principle I.

By repeated application of Principle II we can construct a tree T of depth n with 2^n points of height 1 (or atoms). Call them p_1, \dots, p_{2^n} . A set of atoms q_1, \dots, q_n , are said to be independent if for any q_i and any other q_{j_1}, \dots, q_{j_k} in the set $q_i \not\leq q_{j_1} \vee \dots \vee q_{j_k}$. $\{p_1, \dots, p_{2^n}\}$ contains n points q_1, \dots, q_n generating a boolean lattice of height n . We first demonstrate that $\{p_1, \dots, p_{2^n}\}$ contains n points q_1, \dots, q_n such that $h(q_1 \vee \dots \vee q_n) = n$.

We show this by induction. Set $q_1 = \text{any } p_i \in \{p_1, \dots, p_{2^n}\}$. Then $h(q_1) = 1$.

We prove the induction step as follows. We assume there exists a P_i in T with $h(P_i) = i$ and q_1, \dots, q_i atoms in T with $q_j < P_i \forall j \leq i$. By construction of T there is a P_{i+1} in T with $P_{i+1} > P_i$ and a q_{i+1} , equal to some p_j , with $q_{i+1} < P_{i+1}$, $q_{i+1} \neq q_j \forall j \leq i$, and $q_{i+1} \not\leq P_i$ (if we can't find such a q_{i+1} then

any $p_j < P_{i+1}$ would also be $< P_i$ and, since it follows from the construction of T that P_{i+1} is the join of some set of $p_j < P_{i+1}$ we would have $P_{i+1} \leq P_i$ contradicting the fact that the P_i arose from an application of Principle II to P_{i+1}).

Now $q_{i+1} \not\leq \bigvee_{j \leq i} q_j$ and $h(\bigvee_{j \leq i+1} q_j) > h(\bigvee_{j \leq i} q_j)$. But $h(\bigvee_{j \leq i+1} q_j) \leq h(\bigvee_{j \leq i} q_j) + h(q_j)$. Therefore $h(\bigvee_{j \leq i+1} q_j) = h(\bigvee_{j \leq i} q_j) + 1$.

A set of atoms q_1, \dots, q_n , are said to be independent if for any q_i and any other q_{j_1}, \dots, q_{j_k} in the set $q_i \not\leq q_{j_1} \vee \dots \vee q_{j_k}$.

The lattices in any cover of $M = \{q_1, \dots, q_n\}$ are consistent with the extension of M as a lattice satisfying a height function i.e. a modular lattice. But any set of points $\{q_1, \dots, q_n\}$ in a modular lattice with $q_i \wedge q_j = 0 \forall i, j \leq n$ and $h(\bigvee_{i \leq n} q_i) = n$ are independent and their join, having height n , if it exists is equal to 1. Hence $B_{\{q_1, \dots, q_n\}}$ is unique and Principle III ensures that the structure contains this lattice.

6 Construction Principle IIa

Principles I-IV generate and are realised by the lattice of propositions of classical physics.

As we use each principle to enlarge the structure we choose constants from *Const*, to satisfy the new relation generated by the principle. These new constants need only be related to points already chosen in a manner implied by the relations generated so far and these are the only relations they must satisfy. Therefore there may be many possible choices of constant to substitute in a relation generated by a given principle and hence many distinct possibilities for a structure realising Principles I-IV.

Let us consider a larger structure than one given by Principles I-IV; the structure, call it Q , which comprises all such simultaneously possible structures. What meaning may we attach to Q ? If a structure generated by Principles I-IV represents a physical description of the world then Q would contain all the physical descriptions that are simultaneously possible based on our treatment of analysis in a boolean language. Q then represents the fullest description of the world if the world is restricted to statements which belong in some model of this conception of a physical theory. Q is generated by strengthening Principle II in the following way.

In the definition of Principle II, for a given a in the structure we define elements b_a and c_a , chosen from *Const*, with $b_a \vee c_a = a$. To generate Q we define the stronger Principle IIa which says there also exists $b'_a \in \text{Const}$ distinct from b_a and c_a with $b'_a \vee c_a = a$ and such that $\forall e \ c_a \vee e = c_a$ and $b'_a \vee e = b'_a$ iff $e = 0$.

It is only this principle which must be strengthened to generate Q . The 0 and 1 of Principle I are, by definition, unique and no new points are introduced by Principle IV. The points introduced by Principle III are either uniquely defined by \wedge and \vee from points already in the structure or else generated by taking complements. However we will show that the Principles I, III, IV and the augmented Principle IIa generate a structure in which each point has a non unique complement and so it is not necessary to augment Principle III.

Formally we extend Principle II as follows to generate the structure comprising all possibilities that arise in creating a model in the manner described above.

PRINCIPLE II_A *Let a be a point in the structure to which we may apply the analytic Principle II (subject to Principle IV) . Then in addition to b_a, c_a distinct, with $b_a \vee c_a = a$, there also exists b'_a with b_a, c_a, b'_a all distinct and $b'_a \vee c_a = a$*

Let Q be the structure generated by PRINCIPLES I, IIa, and III, together with the AD HOC PRINCIPLE IV restricting the range of h .

7 Q generates and is realised in the lattice of Quantum Mechanical propositions

We now show Q realises and contains P_{n-1} , the projective lattice of dimension $n - 1$, Birkhoff and Von Neumann's characterisation of the lattice of quantum mechanical propositions. They define P_{n-1} as the modular atomic perspective lattice of height bounded by n consisting of the subspaces of the projective lattice P_{n-1} under set intersection and linear sum.

However the following equivalent characterisation will also be useful.

P_{n-1} is a lattice defined as follows;

Call the atoms of the lattice 'points', the elements λ with $h(\lambda) = 2$ 'lines', and the elements π with $h(\pi) = 3$ 'planes'. We say a point, p , is on a line, λ , when $p \leq \lambda$ in the lattice. For a line, λ , and a plane, π , if $\lambda \leq \pi$ the line is said to lie in the plane.

P_{n-1} is then defined to have the following properties.

- P1** Two distinct points are on one and only one line.
- P2** If two lines lie in the same plane they have a nonempty intersection.
- P3** Every line contains at least three points.
- P3** The set of all points is spanned by n points but not by fewer than

n points. i.e. there is a set of points p_1, \dots, p_n such that for p , a point in P_{n-1} , $p \leq p_1 \wedge p_2, \dots, p_n$.

7.1 P_{n-1} contains Q

Let P_{n-1} be a modular atomic perspective lattice of height bounded by n . We will show P_{n-1} realises any statements constructed according to Principles I, IIa, III and IV (and hence these Principles are consistent).

P_{n-1} obviously realises points 0 and 1 satisfying Principle I.

Let Q , a structure generated by I, IIa, III and IV, be realised in P_{n-1} . Then as in the earlier case, given any a in Q to which Principle IIa can be applied subject to Principle IV, a is realised as an element of height > 1 in P_{n-1} and hence $\exists b_a, c_a \in P_{n-1}$ with $b_a \vee c_a = a$ and $b_a \wedge c_a = 0$ as required. Since P_{n-1} is perspective $\exists b'_a \in P_{n-1}, b'_a \neq b_a, c_a$ with $b'_a \vee c_a = b'_a \vee b_a = a$ and $b'_a \wedge c_a = 0$. Hence P_{n-1} realises this additional extension of Q arising from the application of Principle IIa (subject to Principle IV).

Let M be a substructure of Q realised in P_{n-1} and such that Principle III generates a substructure of a boolean lattice, B_M , containing M and consistent with Principle IV as described above.

Since M is realised in P_{n-1} each point in M may be expressed as a join of atoms in P_{n-1} . Then there is a boolean lattice in P_{n-1} realising a B_M , i.e. there is a set of independent atoms in P_{n-1} , p_1, \dots, p_n generating a boolean lattice in the same relation to M as some B_M .

Suppose such a set does not exist. Let p_1, \dots, p_n be a set of atoms in P_{n-1} generating the points representing M . Further there is no smaller set of atoms with the same property. Then by hypothesis $\exists I, J \subset \{1, \dots, n\} I \cap J = \emptyset$ and $\bigvee_{i \in I} q_i \wedge \bigvee_{j \in J} q_j \neq \emptyset$. Let B_{M_i} be some boolean sublattice of P_{n-1} containing the $q_i, i \in I$. Then the existence of a boolean lattice in P_{n-1} containing M is contradicted by the existence of the $q_i, i \in I$ with the above properties and therefore we have a cover of M that contradicts the condition for the application of Principle III.

Hence P_{n-1} realises an extension of Q by application of Principle III.

In the above P_{n-1} was shown to realise Principles IIa and III subject to Principle IV.

7.2 Q contains P_{n-1}

Here we show that a structure generated by I, IIa, III and IV, satisfies conditions (i)-(iv) defining P_{n-1} . Let Q be such a structure.

(i) For any two distinct atoms $a, b \in Q$ $a \vee b \in Q$.

Let a, b be atoms in Q . In any given cover of $M = [a, b]$ the lattices in the cover are consistent with the existence of $a \vee b$ and $a \wedge b$ satisfying a modular height function i.e. $h(a \vee b) = 2, h(a \wedge b) = 0$. They are also consistent with Principle II by which $\exists c$ with $c \vee (a \wedge b) = 1$ and $\forall e$ $c \vee e = c, e \vee (a \wedge b) = (a \wedge b) \Rightarrow e = 0$. And, as in the earlier case, repeated application of Principle II to c shows the existence of a set of distinct atoms q_1, \dots, q_{n-2} s.t. in a modular lattice $\{q_1, \dots, q_{n-2}, a, b\}$ generate a $B_{\{q_1, \dots, q_{n-2}, a, b\}}$ and so any cover contains such a boolean lattice and Q contains $a \vee b$.

(ii) Every line contains a third point.

Let $a \vee b$ define a line L , a, b , atoms in Q . Then by Principle IIa there exists b' in Q with $a \vee b' = L$.

(iii) Two lines L, L' lying in the same plane P have a non empty intersection.

In any given cover of $M = [L_1, L_2, P]$ the lattices in the cover are consistent with the existence of $L_1 \vee L_2$ and $L_1 \wedge L_2$ with height determined as follows. $h(L_1 \vee L_2) > 2$ since $L_1 \neq L_2$, $h(L_1 \vee L_2) \leq 3$ since $L_1 < P, L_2 < P$ and so $h(L_1 \vee L_2) = 3, L_1 \vee L_2 = P$ and $h(L_1 \vee L_2) = h(L_1) + h(L_2) - h(L_1 \wedge L_2) = 1$. Set $L_1 \wedge L_2 = q'$ then any set of lattices in the cover are consistent with the existence of a pair of points q_1, q_2 not contained in any of the other lattices with $h(q_1) = h(q_2) = 1, q_1 \vee L_1 = L_1, q_2 \vee L_2 = L_2$ and hence since these obey a height function $q_1 \vee q' = L_1, q_2 \vee q' = L_2, q_1 \vee q_2 \vee q_i = P$. Arguing as in (i) we can show the existence of a set of atoms $\{q_4, \dots, q_n\}$ such that any modular lattice containing $\{q_1, q_2, q', q_4, \dots, q_n\}$ contains a $B_{\{L_1, L_2, P\}}$.

Therefore there exists a $B_{\{L_1, L_2, P\}}$ in every cover containing $L_1 \wedge L_2$ with $h(L_1 \wedge L_2) = 1$ and so $h(L_1 \wedge L_2) = 1$ in Q .

(iv) Since Principle IIa implies Principle II and we have already shown Principles I, II, and III subject to Principle IV generate a boolean lattice of height n condition **P3** is satisfied in Q .

8 Conclusion

In this paper we have shown how the exotic modular perspective lattice of quantum mechanical propositions P_{n-1} can be constructed from a more intuitive set of assumptions about a physical theory - in particular our assumptions about what constitutes an analytic procedure and those about

the logical structure of the language in which we deal with this analytic procedure. Forming a model of everything we can construct in this way we generate the lattice of Quantum Mechanical propositions.

Any physical theory contains a notion of measurement founded on analysis or the division of the whole into parts. This idea of ‘part of’ is represented by a partial ordering relation . If this partition arises from some finite process ‘inf’ and ‘sup’ may be taken giving a binary relation \wedge which can, potentially, be extended to a modular complemented lattice.

If the lattice does not contain perspective elements we can construct this extension in actuality as follows. Any natural language in which we treat the partition operator contains the boolean operators \wedge_L and \vee_L and $'_L$ which may be used subject to requirements of consistency to extend the structure generated by the partition operator to a lattice.

However in attempting to construct a representation of partition in this way many points from Set may be chosen to represent the elements of the structure as they are generated - i.e. the elements may be named in a variety of ways.

For a particular set of such choices the partition structure can be extended to a boolean lattice by the boolean operators of the language in the manner described above.

If we consider the structure generated by each such set of choices to be a legitimate set of physical propositions the collection of all simultaneously possible such sets will be the fullest physical description of the world.

However the complemented modular lattice to which the partial order can potentially be expanded now contains perspective elements and hence can not be expanded to a boolean lattice by the boolean operators of the language. Instead we can only embed fragments of this structure representing partition in our boolean languages. The model that is implied by the possibility of expanding these fragments in this way is P_{n-1} .

The central point to the development of P_{n-1} described here is that the expansion in terms of the usual boolean operators of the natural notion of partition, when we take in to account the freedom that arises in naming the elements of the structures, is sufficient to generate the problematic modular operator of P_{n-1} .

What are the consequences of this development of the quantum mechanical lattice of propositions for the interpretation of quantum mechanics ?

Wave particle duality as observed in the two slit experiment is a consequence of P_{n-1} satisfying the perspectivity property derived above. Another consequence of the independence of the perspectivity relation, given by Principle IIa, from the principles generating a boolean lattice is that there exist

propositions in P_{n-1} outside a given boolean lattice in P_{n-1} . This notion of independence may be reformulated to give many of the interpretations of Quantum Mechanics. e.g. technically randomness can be defined in terms of such independence and this leads in a natural way to the probabilistic interpretation of Quantum Mechanics .

What has been done here is to provide from an examination of the nature of physical theories a set of generating principles for physical propositions that are formally stronger than, or independent of, principles generating a given boolean lattice of physical propositions – such formal independence being already the basis for most interpretations of Quantum Mechanics .

9 Bibliography

[1] ‘The logic of Quantum Mechanics’ G.Birkhoff, J. von Neumann *Annals of Mathematics* **37** 1936, 823-43.